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NC Calabi–Yau manifolds in toric varieties with NC torus fibration

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Abstract

Using the algebraic geometry method of Berenstein and Leigh (BL) (hep-th/0009209 and hep-th/0105229), and considering singular toric varieties \mathcal{V}_{d+1} with NC irrational torus fibration, we construct NC extensions $\mathcal{M}_d^{\text{nc}}$ of complex d dimension Calabi–Yau (CY) manifolds embedded in $\mathcal{V}_{d+1}^{\text{nc}}$. We give realizations of the NC \mathbf{C}^{*r} toric group, derive the constraint equations for NC Calabi–Yau (NCCY) manifolds $\mathcal{M}_d^{\text{nc}}$ embedded in $\mathcal{V}_{d+1}^{\text{nc}}$ and work out solutions for their generators. We study fractional D branes at singularities and show that, due to the complete reducibility property of \mathbf{C}^{*r} group representations, there is an infinite number of noncompact fractional branes at fixed points of the NC toric group.

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1. Introduction

Since the original work of Connes et al. on matrix model compactification on noncommutative (NC) torii [1], an increasing interest has been devoted to the study of NC spaces in connection with solitons in NC quantum field [2], and string field theories [1–3]; in particular in the analysis of $D(p-4)/Dp$ brane systems ($p > 3$) of superstrings [4–7] and in the study of tachyon condensation using the GMS approach [8]. However most of the NC spaces considered in these studies involve mainly NC \mathbf{R}_θ^d [9], NC \mathbf{T}_θ^d torii [9,10], some cases of \mathbf{Z}_n type orbifolds of NC torii [11,12] and some generalizations to NC higher-dimensional cycles such as the NC Hizerbruch complex surface F_0

used in [13] and some special CY orbifolds. Quite recently efforts have been devoted to go beyond these particular manifolds by considering particular examples of CY manifolds \mathcal{M} , especially those given by homogeneous hypersurfaces with discrete symmetries [14–18] in projective spaces such as the quintic \mathcal{Q} [19,20]. Such analysis is important for the stringy resolution of singularities; but also for the study of fractional D branes at these singularities [21–23]. The key point in the construction of NCCY orbifolds by help of algebraic geometry method, is based on solving noncommutativity in terms of the torsion of discrete isometries of the orbifolds. This idea was successfully applied in [15], for the study ALE spaces and aspects of the NC quintic; then it has been extended in [17,18] for the building of NC orbifolds $\mathcal{H}_d^{\text{nc}}$ of complex d dimension homogeneous hypersurfaces \mathcal{H}_d . In this Letter, we want to go one step further by extending these

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results to the large class of complex d dimension CY *nonhomogeneous* manifolds \mathcal{M}_d embedded in singular toric varieties \mathcal{V}_{d+1} [25–27]. The organization of this Letter is as follows: in Section 2, we review general aspects of CY manifolds \mathcal{M}_d embedded in toric varieties \mathcal{V}_{d+1} and focus on the study of the type IIA geometry. Sections 3 and 4 are mainly devoted to the study of the NC type IIA extension of \mathcal{M}_d by introducing a NC toric fibration with \mathcal{M}_d as a base. We first show how NC toric fibrations may be realised and then derive and solve the constraint equations for the NC structure. In Section 5, we study fractional branes at singular points and in Section 6, we give our conclusion.

2. CY manifolds in toric varieties

We start by recalling that there are different ways for building complex d dimension CY manifolds M . A tricky way is by embedding M in a toric variety \mathcal{V} ; that is a complex Kahler manifold with some \mathbf{C}^{*r} toric actions. The simplest situation is given by the case where M is described by complex d dimension hypersurface in a complex $d + 1$ toric variety \mathcal{V}_{d+1} . To write down algebraic geometry equations for the CY hypersurfaces which may be singular; one should specify a set of ingredients namely a local holomorphic coordinates patch of the toric manifold \mathcal{V}_{d+1} , the group of toric action and the toric data. To do so, one should moreover distinguish between two kinds of geometries for the CY manifolds \mathcal{M}_d ; (1) the so-called type IIA geometry, to which we will refer to as \mathcal{M}_d , and (2) its dual type IIB geometry often denoted as \mathcal{W}_d . The latter is obtained from \mathcal{M}_d by exchanging their Kahler and complex structures following from the Hodge identities $h^{1,1}(\mathcal{M}_d) = h^{d-1,1}(\mathcal{W}_d)$ and $h^{1,1}(\mathcal{W}_d) = h^{d-1,1}(\mathcal{M}_d)$ [28]. In this study, we will mainly focus on the type IIA geometry.

A nice way to introduce the type IIA geometry \mathcal{M}_d is in terms of 2D $N = 2$ field theory which is known to describe the propagation of type IIA strings on \mathcal{M}_d . In this approach, the \mathcal{M}_d background is constructed in terms of $N = 2$ supersymmetric linear sigma models involving a superfield system $\{V_a, X_i\}$ containing r gauge $N = 2$ Abelian multiplets $V_a(\sigma, \theta, \bar{\theta})$ with gauge group $U(1)^r$ and $k + 1$ chiral matter superfields $X_i(\sigma, \theta, \bar{\theta})$ of bosonic com-

ponents x_i . In addition to the usual kinetic terms and gauge-matter couplings, the linear sigma model action $S[V_a, X_i]$ of these fields may have r Fayet–Iliopoulos (FI) D-terms, which in the language of $N = 2$ superfields read as $\zeta_a \int d^2\sigma d^2\theta d^2\bar{\theta} V_a(\sigma, \theta, \bar{\theta})$, with ζ_a being the FI coupling constants. The superfields action $S[V_a, X_i]$ may also have a holomorphic superpotential $W(X_0, \dots, X_k)$ given by polynomials in the X_i 's, which in the infrared limit, is known to describe the CFT₂ of string propagation on the type IIA background [29]. Let us discuss a little bit this special geometry.

In the method of toric geometry, where to each complex bosonic field x_i it is associated some toric data $\{q_i^a, \mathbf{v}_i\}$, or more generally by taking into account relevant data from the dual geometry $\{q_i^a, \mathbf{v}_i; p_\alpha^l, \mathbf{v}_\alpha^*\}$ [19,20,25], with \mathbf{v}_i and \mathbf{v}_α^* being $d + 1$ dimension vectors of \mathbf{Z}^{d+1} self dual lattice, one can write down the algebraic geometry equation of the complex d CY manifold \mathcal{M}_d . This is given by a holomorphic polynomial in the x_i 's with some Abelian complex symmetries. In the simplest situation where the toric manifold is given by the coset¹ $\mathcal{V}_{d+1} = \mathbf{C}^{k+1}/\mathbf{C}^{*r}$, $d = k - r$, the complex d dimension CY hypersurface reads as

$$P_d[x_0, \dots, x_k] = b_0 \prod_{i=0}^k x_i + \sum_{\alpha} b_{\alpha} \prod_{i=0}^k x_i^{n_{\alpha i}}, \quad (1)$$

where the b_{α} 's are complex numbers and where the $n_{\alpha i}$ powers are some positive integers constrained by the \mathbf{C}^{*r} invariance. Indeed, under the \mathbf{C}^{*r} , $x_i \rightarrow x_i \lambda_i^{q_i^a}$ with q_i^a some integers, invariance of $P_d[x_0, \dots, x_k]$ requires the $n_{\alpha i}$ integers to be such that

$$\sum_i q_i^a n_{\alpha i} = 0, \quad \sum_i q_i^a = 0. \quad (2)$$

Equations $\sum_i q_i^a = 0$, which ensures the vanishing of the first Chern class of \mathcal{M}_d , follow from the \mathbf{C}^{*r} symmetry of the $\prod_{i=0}^k x_i$ monomial while $\sum_i q_i^a n_{\alpha i} = 0$ come from invariance of $\prod_{i=0}^k x_i^{n_{\alpha i}}$ monomials. Setting $u_{\alpha} = \prod_{i=0}^k x_i^{n_{\alpha i}}$, Eq. (1) can be rewritten as

¹ Toric manifolds are generally defined by cosets $(\mathbf{C}^{k+1} - U)/\mathbf{C}^{*r}$ where $U \subset \mathbf{C}^{k+1}$ is defined by the \mathbf{C}^{*r} actions and a chosen a triangulation. In toric geometry, elements of U are defined by those subsets of vertices, which do not lie together in a single toric cone.

$P_d[u_\alpha] = \sum_\alpha b_\alpha u_\alpha$, where the u_α 's are the effective local coordinates of the coset space $\mathbf{C}^{k+1}/\mathbf{C}^{*r}$. As the u_α 's are given by $u_\alpha = \prod_{i=0}^k x_i^{n_{\alpha i}}$, it may happen that not all of the u_α 's are independent variables; some of them, say u_{α_I} for $I = 1, \dots, r^*$, are expressed in terms of the other u_{α_J} variables with $J \neq I$. In other words; one may have relations type $\prod_\alpha u_\alpha^{p_\alpha^I} = 1$, where p_α^I are some integers. Substituting $u_\alpha = \prod_{i=0}^k x_i^{n_{\alpha i}}$ back into $\prod_\alpha u_\alpha^{p_\alpha^I} = 1$, we discover extra constraint equations on the $n_{\alpha i}$ and p_α^I integers namely

$$\sum_\alpha p_\alpha^I n_{\alpha i} = 0. \quad (3)$$

In toric geometry, the $n_{\alpha i}$ integers are realized as $n_{\alpha i} = \langle \mathbf{v}_i, \mathbf{v}_\alpha^* \rangle = \sum_A v_i^A v_{\alpha A}^*$. In this representation, Eqs. (2), (3) are automatically solved by requiring the toric data of the CY manifold to be such that

$$\sum_i q_i^a \mathbf{v}_i = 0, \quad \sum_\alpha p_\alpha^I \mathbf{v}_\alpha^* = 0. \quad (4)$$

Let us illustrate these relations for the case of the asymptotically local Euclidean (ALE) space with A_{n-1} singularity. This is a complex two dimension $\mathbf{C}^{n+1}/\mathbf{C}^{*(n-1)}$ toric variety with a $su(n)$ singularity $u_0^{-n} u_1 u_2 = 1$ at the origin. From this relation, one sees that there are three invariant variables u_0, u_1 and u_2 ; but only two of them are independent. Since $r^* = 1$, there is only one p_α^I vector of entries $p_\alpha = (-n, 1, 1)$ and three \mathbf{v}_α^* vectors given by $\mathbf{v}_0^* = (1, 0)$, $\mathbf{v}_1^* = (n, -1)$ and $\mathbf{v}_2^* = (0, 1)$. More generally, we have the following cases. (a) No relation such as $\sum_\alpha p_\alpha^I \mathbf{v}_\alpha^* = 0$ exist, that is all the u_α 's independent; the algebraic equation of the hypersurface \mathcal{M}_d is nonsingular and reads as $\sum_{\alpha=0}^d b_\alpha u_\alpha = 0$ with $d = (k - r)$. (b) Generic cases where there exists r^* constraint equations of type $\sum_\alpha p_\alpha^I \mathbf{v}_\alpha^* = 0$, the $(d + r^* + 1)$ complex variables u_α are not all of them independent; so the algebraic geometry equations defining the singular hypersurface embedded in \mathcal{V}_{d+1} is

$$\sum_{\alpha=0}^{d+r^*} b_\alpha u_\alpha = 0, \quad \prod_{\alpha=0}^{d+r^*} u_\alpha^{p_\alpha^I} = 1, \quad I = 1, \dots, r^*, \quad (5)$$

where p_α^I are the integers in Eqs. (4). Note that this hypersurface has singularities at the $u_\alpha = 0$; but the introduction of NC geometry lifts this degeneracy. In

the next section, we will study the NC extension of this geometry.

3. NC type IIA geometry vs NC torus fibration

To start note that NCCY manifolds as built in [15] may be viewed as a NC torus fibration with a CY base. Since NC torii have two kinds of realizations namely rational and irrational representations, we have to distinguish two cases of NC torus fibrations. The first kind of these manifolds involve fuzzy torii and is exactly the solution obtained in [15]. The second class of NC varieties, which extend naturally the previous one, is the type of solution we want to present here. Before that we will first recall the BL algebraic geometry method; then we present our solution by direct use of NC fibration ideas.

3.1. Rational torus

From the algebraic geometry point of view, the NC extension $\mathcal{M}_d^{\text{nc}}$ of the CY manifold \mathcal{M}_d , embedded in \mathcal{V}_{d+1} , is covered by a finite set of holomorphic operator coordinate patches $\mathcal{O}_{(\alpha)} = \{Z_i^{(\alpha)} : 1 \leq i \leq k, \alpha = 1, 2, \dots\}$ and holomorphic transition functions mapping $\mathcal{O}_{(\alpha)}$ to $\mathcal{O}_{(\beta)}$; $\phi_{(\alpha, \beta)} : \mathcal{O}_{(\alpha)} \rightarrow \mathcal{O}_{(\beta)}$. This is equivalent to say that $\mathcal{M}_d^{\text{nc}}$ is covered by a collection of NC local algebras $\mathcal{M}_{d(\alpha)}^{\text{nc}}$ generated by the analytic coordinate of the $\mathcal{O}_{(\alpha)}$ patches of $\mathcal{M}_d^{\text{nc}}$, together with analytic maps $\phi_{(\alpha, \beta)}$ on how to glue $\mathcal{M}_{d(\alpha)}^{\text{nc}}$ and $\mathcal{M}_{d(\beta)}^{\text{nc}}$. The $\mathcal{M}_{d(\alpha)}^{\text{nc}}$ algebras have centers $\mathcal{Z}_{(\alpha)} = \mathcal{Z}(\mathcal{M}_{d(\alpha)}^{\text{nc}})$; when glued together give precisely the commutative manifold \mathcal{M}_d . In this way, a singularity of $\mathcal{M}_d \simeq \mathcal{Z}(\mathcal{M}_d^{\text{nc}})$ can be made smooth in the noncommutative space $\mathcal{M}_d^{\text{nc}}$ [18,24]. This idea was successfully used in the building of NC extensions of the CY homogeneous hypersurface $z_1^{d+2} + z_2^{d+2} + z_3^{d+2} + z_4^{d+2} + z_5^{d+2} + a_0 \prod_{\mu=1}^{d+2} z_\mu = 0$ having \mathbf{Z}_{d+2}^d discrete symmetries acting as $z_i \rightarrow z_i \cdot \omega^{q_i^a}$, where the q_i^a integers satisfy the CY condition $\sum_{i=1}^{d+2} q_i^a = 0$, $a = 1, \dots, d$; [15,16]. The NC $\mathcal{H}_d^{\text{nc}}$ extending \mathcal{H}_d were shown to be given, in the coordinate patch $Z_{d+2} \sim I_{\text{id}}$, by the following NC algebra

$$\begin{aligned} Z_i Z_j &= \theta_{ij} Z_j Z_i, \quad i, j = 1, \dots, (d+1), \\ Z_i Z_{d+2} &= Z_{d+2} Z_i, \quad i = 1, \dots, (d+1). \end{aligned} \quad (6)$$

The θ_{ij} parameters are solved by discrete torsion as $\theta_{ij} = \omega_{ij} \bar{\omega}_{ji}$ with $\bar{\omega}_{kl}$ the complex conjugate of ω_{kl} . The ω_{ij} 's are realized in terms of the q_i^a CY charges and the \mathbf{Z}_{d+2}^d discrete group elements $\omega = \exp i \frac{2\pi}{d+2}$ as $\omega_{ij} = \exp i (\frac{2\pi}{d+2} m_{ab} q_i^a q_j^b) = \omega^{m_{ab} q_i^a q_j^b}$ with m_{ab} integers. The Z_i 's form a regular representation of the discrete group and are solved as

$$Z_i = x_i \prod_{a,b=1}^d \mathbf{P}^{m_{ab} q_i^a} \mathbf{Q}_a^{q_i^b}, \quad (7)$$

where $\mathbf{P} = \text{diag}(1, \omega, \dots, \omega^{d+1})$ and \mathbf{Q} satisfy

$$\mathbf{P}^{d+2} = \mathbf{Q}^{d+2} = I_{\text{id}}, \quad (8)$$

$$\mathbf{P}\mathbf{Q} = \omega \mathbf{Q}\mathbf{P}, \quad \omega = \exp i \frac{2\pi}{d+2}, \quad (9)$$

and generate the NC rational torus. From this solution, one sees that the BL manifold $\mathcal{H}_d^{\text{nc}}$ is just a kind of a NC torus fibration based on the commutative CY hypersurface \mathcal{H}_d .

3.2. Irrational torus

To build the NC type IIA geometry extending Eq. (1), we will mainly adopt the same method as in [15,17] and look for realizations of the NC variables Z_i as

$$Z_i \sim x_i \prod_{a=1}^r U_a^{r_i^a} V_a^{s_i^a}, \quad (10)$$

where r_i^a and s_i^a are positive integers to be determined later. In this fibration, U_a and V_b are the generators of a complex r dimension NC torus satisfying, amongst others, the familiar relation

$$U_a V_b = \gamma_{ab} V_b U_a \quad (11)$$

with γ_{ab} some given irrational C -numbers. One of the main differences between this fibration and the rational torus one of [15]; see also Eqs. (7)–(9), is that here γ_{ab} is no longer a root of unity and so the Z_i 's have infinite-dimensional representations which make such kind of NC extension very special as we will see when we discuss fractional D branes. For the moment let us expose our result by first giving the constraint equations, their solutions and the Z_i 's regular representations.

3.2.1. Constraint equations

Extending naively the algebraic geometry method used for $\mathcal{H}_d^{\text{nc}}$ to our present case by associating to each x_i variable the operator Z_i , then taking $q_k^a = 0$ and working in the coordinate patch $Z_k = I_{\text{id}}$, the NC type IIA geometry $\mathcal{M}_d^{\text{nc}}$ may be defined as

$$\begin{aligned} Z_i Z_j &= \theta_{ij} Z_j Z_i, \quad i, j = 0, \dots, k, \\ Z_i Z_k &= Z_k Z_i. \end{aligned} \quad (12)$$

Since \mathcal{M}_d should be in the centre of $\mathcal{M}_d^{\text{nc}}$, it follows that the Z_i generators and the θ_{ij} parameters should satisfy the constraint equations,

$$\begin{aligned} \left[Z_i, \prod_{j=0}^k Z_j^{n_{\alpha j}} \right] &= 0, \\ \prod_{j=0}^k \theta_{ij}^{n_{\alpha j}} &= 1, \quad \forall i, \theta_{ij} \theta_{ji} = 1. \end{aligned} \quad (13)$$

Actually these relations constitute the defining conditions of NC type IIA geometry $\mathcal{M}_d^{\text{nc}}$. While the constraint relation $\theta_{ij} \theta_{ji} = 1$ shows that $\theta_{ij} = \theta_{ji}^{-1}$, non-trivial solution of the constraint equations $\prod_{j=0}^k \theta_{ij} = 1$ are expressed in terms of $\{q_i^a; v_i^A; p_\alpha^I; v_{\alpha A}^*\}$ data.

3.2.2. Solving the constraint equations

First of all note that since the θ_{ij} 's are nonzero parameters, one may set

$$\begin{aligned} \theta_{ij} &= \prod_{a,b=1}^r \eta_{ab}^{J_{ij}^{ab}}, \\ \eta_{ab} &= \exp(\beta_a \beta_b), \quad \beta_a \in \mathbf{C}, \end{aligned} \quad (14)$$

and solve the constraint Eqs. (13) by introducing torsion for the \mathbf{C}^{*r} toric actions. Putting these parameterizations back into Eqs. (13), one gets the following constraint on the J_{ij}^{ab} 's,

$$\sum_{i=0}^k J_{ij}^{ab} = 0, \quad J_{ij}^{ab} = -J_{ji}^{ab}. \quad (15)$$

Moreover as we are looking for solutions to the Z_i operators as in Eq. (10), let us explore what one may call NC \mathbf{C}^{*r} toric actions.

NC \mathbf{C}^{*r} toric group Recall the \mathbf{C}^{*r} toric group as used in toric geometry is a complex Abelian group which reduce to $U(1)^r$ once the group elements $\exp i\psi_a T_a$, with parameters $\psi_a = \alpha_a - i\rho_a \in \mathbf{C}$, are chosen as $\exp i\alpha_a Q_a$ where the α_a 's are now real numbers. To have NC toric actions, there are two ways to do: (1) either through the use quantum symmetries as in case of discrete torsion or (2) by introducing CDS noncommutative torii [1]. Let us comment briefly these two realizations.

1. Quantum toric symmetry To illustrate the idea, we consider the simple case $r = 1$. Since \mathbf{C}^* is an Abelian continuous group and its representations have very special features, we have to distinguish the usual cases: (a) the discrete infinite-dimensional spectrum and (b) the continuous one. Both of these realizations are important for the present study and should be thought of as extensions of the rational and irrational representations of NC real torii [10,12].

1.a. Discrete spectrum Let $\mathcal{R}_{\text{dis}}(\mathbf{C}^*) = \{U = \exp i\psi T\}$ denote a representation of \mathbf{C}^* on an infinite-dimensional space \mathbf{E}_{dis} with a discrete spectrum generated by the orthonormal vector basis $\{|n\rangle: n = (n_1, n_2) \in \mathbf{Z} \times \mathbf{Z} \sim \mathbf{Z}^2\}$.² Here ψ is a complex parameter, $\psi \in \mathbf{C}$; and T is the complex generator of \mathbf{C}^* ; the ψT combination may be split as $\psi T = \rho K + i\alpha Q$, where K is the generator of dilatations and Q is the generator of phases. For $\psi = \alpha$ real, $\mathcal{R}_{\text{dis}}(\mathbf{C}^*)$ reduces to $\mathcal{R}_{\text{dis}}(U(1)) = \{U = \exp i\alpha Q\}$. The generator T of $\mathcal{R}_{\text{dis}}(\mathbf{C}^*)$ acts diagonally on the vector basis $\{|n\rangle\}$; i.e., $T|n\rangle = n|n\rangle$ and so $U|n\rangle = (\exp i\psi n)|n\rangle$ or equivalently $U = \sum_n \chi_n(\psi) \pi_n$, with characters $\chi_n(\psi) = \exp(i\psi n)$ and $\pi_n = |n\rangle\langle n|$, the projectors on $|n\rangle \pi_n$. Since $\mathcal{R}_{\text{dis}}(\mathbf{C}^*)$ is completely reducible, its I_{id} identity operator may be decomposed in a series of π_n as

$$I_{\text{id}} = \sum_n \pi_n. \quad (16)$$

Like for $U(1)$ and \mathbf{Z}_N subgroups, \mathbf{C}^* has also a complex shift operator V_τ acting on $\{|n\rangle: n \in \mathbf{Z} + i\mathbf{Z}\}$ as an automorphism exchanging the characters $\chi_n(\psi)$. This translation operator which operate as $V_\tau|n\rangle =$

$|n + \tau\rangle$; with $\tau = 1 + i$, may also be defined by help of the $\mathbf{a}_{(n_1, n_2)}^+ = |(n_1 + 1) + in_2\rangle\langle n_1 + in_2|$ and $\mathbf{b}_{(n_1, n_2)}^+ = |n_1 + i(n_2 + 1)\rangle\langle n_1 + in_2|$ step operators as $V_\tau = V_1 \otimes V_i$ with

$$V_1 = \sum_{n_1, n_2 \in \mathbf{Z}} \mathbf{a}_{(n_1, n_2)}^+, \quad V_i = \sum_{n_1, n_2 \in \mathbf{Z}} \mathbf{b}_{(n_1, n_2)}^+. \quad (17)$$

Due to the remarkable properties $\mathbf{a}_{(n_1, n_2)}^+ \pi_{(n_1, n_2)} = \pi_{(n_1+1, n_2)} \mathbf{a}_{(n_1, n_2)}^+$, $\mathbf{b}_{(n_1, n_2)}^+ \pi_{(n_1, n_2)} = \pi_{(n_1, n_2+1)} \mathbf{b}_{(n_1, n_2)}^+$ and $\mathbf{b}_{(n_1, n_2)}^+ \mathbf{a}_{(n_1, n_2)}^+ \pi_{(n_1, n_2)} = \pi_{(n_1+1, n_2+1)} \mathbf{b}_{(n_1, n_2)}^+ \times \mathbf{a}_{(n_1, n_2)}^+$, it follows that U and V satisfy the following noncommutative algebra

$$UV = e^{-i\psi\tau} VU, \quad (18)$$

describing the complex extension of the CDS torus [1]; to which we shall refer to as the NC complex two torus. Since ψ is an arbitrary complex parameter, Eqs. (18) define an irrational NC complex two torus.

1.b. Continuous case In this case, the generator T of $\mathcal{R}_{(\text{con}, \text{con})}(\mathbf{C}^*)$ has a continuous spectrum with a vector basis $\{|z\rangle, z \in \mathbf{C}: \langle z'|z\rangle = \delta(z - z')\}$. It acts diagonally as $\langle z|T|z\rangle = z\langle z|$ and so $\langle z|U = (\exp i\psi z)\langle z|$ which imply in turns, $U = \int dz \chi(\psi, z) \pi(z)$ with $\chi(\psi, z) = \exp(i\psi z)$ and $\pi(z) = |z\rangle\langle z|$; $\pi(z)\pi(z') = \delta(z - z') \pi(z)$. Since $\mathcal{R}_{(\text{con}, \text{con})}(\mathbf{C}^*)$ is completely reducible, the I_{id} identity operator may be decomposed as

$$I_{\text{id}} = \int dz \pi(z). \quad (19)$$

The shift operator by an ϵ amount, denoted as $V(\epsilon)$, acts on $\{|z\rangle, z \in \mathbf{C}\}$ as $\langle z|V(\epsilon) = \langle z + \epsilon|$. It may be defined, by help of $\mathbf{a}^+(z, \epsilon) = |z\rangle\langle z + \epsilon|$ operators, as

$$V(\epsilon) = \int dz \mathbf{a}^+(z, \epsilon). \quad (20)$$

These operators satisfy similar relations as in Eqs. (17), (18) namely $UV = \exp(-i\psi\tau) VU$ and may also be realized on the space of holomorphic functions $F(z) = \langle z|F\rangle$ as

$$UFU^{-1} = (\exp i\psi z)F, \\ V_\epsilon F V_\epsilon^{-1} = (\exp \epsilon \partial_z)F. \quad (21)$$

We give details on the differential realization of U and V operators in the next section.

² We will use the convention notation $n \equiv n_1 + in_2 \in \mathbf{Z} + i\mathbf{Z} \sim (n_1, n_2) \in \mathbf{Z}^2$; $|n\rangle$ should be then thought of as $|n_1\rangle \otimes |n_2\rangle$.

2. NC \mathbf{C}^* toric cycles If one forgets for a while about the quantum symmetry generated by V_a and focus on the \mathbf{C}^* toric generators U_a only, one may also build representations where the r complex cycles of the \mathbf{C}^{*r} group are noncommuting. This is achieved by demanding $[T_a, T_b] = im_{[ab]} \neq 0$ which is ensured by introducing torsion among the \mathbf{C}^* factors. Here also we should distinguish between discrete and continuous spectrums. In the continuous case for instance, the algebra of NC toric cycles is

$$U_a U_b = \Lambda_{ab}^{m_{[ab]}} U_b U_a, \quad \Lambda_{ab} = \exp(-i\psi_a \psi_b),$$

$$[T_a, T_b] = im_{[ab]} I_{\text{Id}}, \quad (22)$$

where m_{ab} is a $r \times r$ matrix. A realization of T_a on the space \mathcal{F} of holomorphic functions $f(y_1, \dots, y_r)$ with r arguments, is given by

$$[T_a, f(y_1, \dots, y_r)] = (\partial_a - im_{ac} y_c) f. \quad (23)$$

Quantum symmetries may be also considered by allowing the $f(y_1, \dots, y_r)$ functions to depend on extra variables z_a so that we have new function $F(z_a; y_a)$ satisfying

$$U_a F U_a^{-1} = (\exp i\psi_a (z_a + \partial_{y_a} - im_{ac} y_c)) F,$$

$$V_b F V_b^{-1} = (\exp \epsilon_{bd} \partial_{z_d}) F. \quad (24)$$

Having studied the main lines of NC toric actions, we turn now to solve the constraint Eqs. (13)–(15).

4. Representations of the Z_i 's

The constraint Eqs. (13) may be solved in different ways depending on whether quantum symmetries are taken into account or not. In general, τ_{ab} torsion between U_a and V_a generators is introduced through the relations $U_a V_b = \Omega_{ab}^{\tau_{ab}} V_b U_a$. The solution for the Z_i 's reads as

$$Z_i = x_i \prod_{a=1}^r \exp i \tilde{q}_i^a (\psi_a T_a + \phi_a S_a), \quad (25)$$

where the T_a and S_a operators may be realized, for the case of a continuous spectrum, as

$$T_a = \frac{\partial}{\partial y^a} + z_a - im_{ac} y_c, \quad S_a = \epsilon^{ab} \frac{\partial}{\partial y^b} \quad (26)$$

and where x_i are complex moduli, which we shall interpret as just the commutative coordinates of the

toric manifold \mathcal{V}_{d+1} . The \tilde{q}_i^a 's are shifted CY charges having extra contributions coming from the toric data, $\tilde{q}_i^a = q_i^a + \sum_{A=1}^d k_A Q_i^{aA} + \sum_{\alpha} k^{\alpha} N_{i\alpha}^a$ with k_A and k^{α} are numbers; they still obey $\sum_{i=0}^k \tilde{q}_i^a = 0$ which follow from the identities $\sum_{i=0}^k Q_i^{aA} = \sum_{i=0}^k N_{i\alpha}^a = 0$ Eqs. (4). The θ_{ij} parameters of the NC type IIA geometry we get are

$$\theta_{ij} = \prod_{a,b=1}^r \Lambda_{ab}^{J_{ij}^{ab}} \Omega_{ab}^{K_{ij}^{ab}}, \quad (27)$$

where now $J_{ij}^{ab} = m_{[ab]} \tilde{q}_i^a \tilde{q}_j^b$ and $K_{ij}^{ab} \sim \tau_{[ab]} \tilde{q}_i^a \tilde{q}_j^b$. By appropriate choices of Λ_{ab} , Ω_{ab} , $m_{[ab]}$ and $\tau_{[ab]}$, one recovers, as special cases, the representations involving discrete torsions obtained in [15,17]. According to the nature of spectrums of T_a and S_a , the Z_i operators will have two sectors; discrete and continuous. To illustrate the previous analysis, we consider the NC extension of a CY manifold with a conic singularity. This manifold is defined by a hypersurface \mathcal{M}_2 embedded in the toric variety $\mathcal{V}_3 \subset \mathbf{C}^6 / \mathbf{C}^{*2}$ with \mathbf{C}^{*2} actions $x_i \rightarrow x_i \exp i(\psi_a q_i^a)$ where $q_i^1 = (1, -1, 1, -1, 0, 0)$, $q_i^2 = (1, 0, -1, 1, -1, 0)$ and $p_{\alpha} = (1, -1, 1, -1)$. The $v_i^A = (v_i^1, v_i^2, v_i^3, v_i^4)$ and $v_{iA}^* = (v_{i1}^*, v_{i2}^*, v_{i3}^*, v_{i4}^*)$ vertices satisfy $\sum_{i=1}^6 q_i^a v_i = 0$ and $\sum_{\alpha=1}^4 p_{\alpha} v_{\alpha}^* = 0$,

$$v_i = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix},$$

$$v_{\alpha}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}. \quad (28)$$

The four u_{α} gauge invariants read as $u_{\alpha} = \prod_{i=1}^6 x_i^{n_{\alpha i}}$, $i = 1, \dots, 6$, with $n_{\alpha i}$ integers as

$$n_{\alpha i} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 & 2 & 2 \\ 3 & 3 & 0 & 0 & 3 & 5 \\ 2 & 2 & 1 & 1 & 2 & 4 \end{pmatrix}. \quad (29)$$

They satisfy the constraint equation $n_{0i} + n_{2i} = n_{1i} + n_{3i}$, showing that \mathcal{V}_3 is described by $u_0 u_2 = u_1 u_3$ with a conic singularity at the origin. The complex two dimension CY hypersurface embedded in \mathcal{V}_3 , reads,

in terms of the x_i local coordinates, as

$$P(x_1, \dots, x_6) = ax_1^2x_2^2x_5^2x_6^2 + bx_1^3x_2^3x_5^3x_6^5 + cx_1^2x_2^2x_3x_4x_5^2x_6^4 + d \prod_{i=1}^6 x_i. \quad (30)$$

The NC extension $\mathcal{M}_2^{\text{nc}}$ of this holomorphic hypersurface is directly obtained. For the special case where the L_{ij} antisymmetric matrix is restricted to $L_{ij} = m(q_i^1q_j^2 - q_j^1q_i^2)$, with $L_{ij} = -L_{ji}$ and $L_{i6} = 0$ and

$$L_{ij} = m \begin{pmatrix} 0 & 1 & -2 & 2 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 2 & -1 & 0 & 0 & -1 \\ -2 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}, \quad (31)$$

the NC complex surface $\mathcal{M}_2^{\text{nc}}$ is then given by a one parameter algebra generated by following relations,

$$\begin{aligned} Z_1Z_2 &= \Lambda^m Z_2Z_1, & Z_1Z_3 &= \Lambda^{-2m} Z_3Z_1, \\ Z_1Z_4 &= \Lambda^{2m} Z_4Z_1, & Z_1Z_5 &= \Lambda^{-m} Z_5Z_1, \\ Z_2Z_3 &= \Lambda^m Z_3Z_2, & Z_2Z_4 &= \Lambda^{-m} Z_4Z_2, \\ Z_2Z_5 &= \Lambda^m Z_5Z_2, & Z_3Z_4 &= Z_4Z_3, \\ Z_3Z_5 &= \Lambda^{-m} Z_5Z_3, & Z_4Z_5 &= \Lambda^m Z_5Z_4, \\ Z_iZ_6 &= Z_6Z_i, \end{aligned} \quad (32)$$

where Λ^m is given by $\Lambda^m = \exp(-im\psi_1\psi_2)$. Since $\psi_a = \rho_a - i\alpha_a$; it follows that $m\psi_1\psi_2 = m(\rho_1\rho_2 - \alpha_1\alpha_2) - im(\alpha_1\rho_2 + \alpha_2\rho_1)$ which we set as $\Lambda^m = \exp(\kappa + i\phi)$ for simplicity. This is a one complex parameter algebra enclosing various special situations corresponding to: (1) *hyperbolic representation* described by $(\kappa, \phi) \equiv (\kappa, 0)$; (2) *periodic representations* corresponding to $(\kappa, \phi) \equiv (0, \phi + 2\pi)$ where $|\Lambda^m| = 1$; (3) *discrete periodic representations* $(\kappa, \phi) \equiv (0, N\phi + 2\pi)$ with $|\Lambda^m| = 1$ but moreover $(\Lambda^m)^N = 1$. This last case is naturally a subspace of the periodic representation and it is precisely the kind of situation that happens in the building of BL NC manifolds with discrete torsion.

5. Fractional branes

The NC type IIA realization we have studied concerns regular points of the algebra. In this section,

we want to complete this analysis by considering the representations at fixed points where live fractional D branes. To do so, we shall first classify the various subspaces $\mathcal{S}_{(\mu)}$ of stable points under \mathbf{C}^{*r} ; then we give the quiver diagrams extending those of Berenstein and Leigh [24].

5.1. Fixed points

The local holomorphic coordinates $\{x_i \in \mathbf{C}^{k+1}; 0 \leq i \leq k\}$ Eq. (1) are not all of them independent as they are related by the \mathbf{C}^{*r} gauge transformations $U_a : x_i \rightarrow U_a x_i U_a^{-1} = x_i \lambda_i^{q_i^a}$, with $\sum_{i=0}^k q_i^a = 0$. Fixed points of the \mathbf{C}^{*r} gauge transformations are given by the solutions of the constraint equation

$$x_i = U_a x_i U_a^{-1}. \quad (33)$$

Solutions of this equation depend on the values of q_i^a ; the x_i 's should be zero unless $q_i^a = 0$. Fixed points are then given by the \mathcal{S} subspace of \mathbf{C}^{k+1} whose x_i local coordinates are \mathbf{C}^{*r} gauge invariants. To get a more insight into this subspace it is interesting to note that as $\mathbf{C}^{k+1}/\mathbf{C}^{*r} = (\mathbf{C}^{k+1}/\mathbf{C}^*)/\mathbf{C}^{*r-1}$; it is useful to introduce the $\mathcal{S}_{(a)} = \{x_i \mid q_i^a = 0, i \in J \subset \{0, 1, \dots, k\}\}$ subspaces that are invariant under the a th factor of the \mathbf{C}^{*r} group. So the manifold \mathcal{S} stable under the full \mathbf{C}^{*r} is given by the intersection of the various $\mathcal{S}_{(a)}$'s, i.e.,

$$\mathcal{S} = \bigcap_{a=1}^r \mathcal{S}_{(a)}. \quad (34)$$

If we denote by $\{x_{i_0}, \dots, x_{i_{k_0-1}}\}$ those local coordinates that have nonzero q_i^a charges; $J = \{i_0, \dots, i_{k_0}\}$, and $\{x_{i_{k_0-1}}, \dots, x_{i_k}\}$ the coordinates that are fixed under \mathbf{C}^{*r} ; then the manifold \mathcal{S} is given by

$$\mathcal{S} = \{(0, \dots, 0, x_{k_0}, \dots, x_k)\} \subset \mathcal{V}_{d+1} \subset \mathbf{C}^{k+1}. \quad (35)$$

To get the representation of the Z_i variable operators on \mathcal{S} , let us first consider what happens on its neighboring space $\mathcal{S}^\epsilon = \{(\epsilon, \dots, \epsilon, x_{k_0} + \epsilon, \dots, x_k + \epsilon)\}$, where ϵ is as small as possible. Using the hypothesis $q_{i_{k_0}}^a = \dots = q_{i_k}^a = 0$ we have made and replacing the x_i moduli by their expression on \mathcal{S}^ϵ , then putting in the realization Eqs. (20), we get the follow-

ing result:

$$Z_{ij} = \epsilon \prod_{a=1}^r \exp i q_i^a (\psi_a T_a + \phi_a S_a),$$

$$0 \leq j \leq k_0 - 1, \quad (36)$$

$$Z_{ij} = (x_{ij} + \epsilon) I_{\text{id}}, \quad k_0 \leq j \leq k. \quad (37)$$

The representation of the Z_i 's on \mathcal{S} is then obtained by taking the limit $\epsilon \rightarrow 0$. As such nontrivial operators are given by $Z_{ij} \sim x_{ij} I_{\text{id}}$, $k_0 \leq j \leq k$; they are proportional to the I_{id} operator of $\mathcal{R}(C^{*r})$. This an important point since the Z_i operators are reducible into an infinite component sum as shown here below³

$$Z_i = \sum_{\mathbf{n}} Z_i^{(\mathbf{n})}, \quad Z_i^{(\mathbf{n})} = x_i \pi_{\mathbf{n}}. \quad (38)$$

5.2. Brane interpretation

The above decomposition of the Z_i 's on \mathcal{S} has a nice interpretation in the D brane language. Thinking about x_i 's as the coordinates of a Dp brane ($p = 2d$), wrapping M_d , it follows that, due NC torus fibration, Dp at singular points fractionate. In addition to the results of [24], which apply as well for the present study, there is a novelty here due to the dimension of the completely reducible $\mathcal{R}(C^{*r})$ representation. There are infinitely many values for the C^* characters and so an infinite number of fractional $D(p - 2k_0)$ branes wrapping \mathcal{S} . However, this is a unacceptable solution from string compactification point of view. Branes should have finite tensions in string theory background unless there are noncompact dimensions. In fact this is exactly what we have since the irrational representation of the NC torus fibration introduces extra noncompact directions as shown on the realization

Eqs. (25), (26). Such behaviour has no analogue in the BL case.

5.3. Quiver diagrams

Like for the BL case of CY orbifolds, one can here also describe the varieties of fractional D branes by generalizing the BL quiver diagrams for $\mathcal{M}_d^{\text{nc}}$ at fixed points. One of the basic ingredients in getting these graphs is the identification of the projectors of \mathbf{C}^{*r} and the step operators \mathbf{a}^{\pm} . Since each \mathbf{C}^* factor has completely reducible representations with four sectors, it is interesting to treat them separately. The various sectors for each \mathbf{C}^* subsymmetry factor are as follows; (i) (dis, dis) discrete–discrete sector where the C^* characters are given by $\chi_n(\psi) = \exp i \psi n$; $n \in \mathbf{Z} + i\mathbf{Z}$, $\psi \in \mathbf{C}$; (ii) (dis, con) discrete–continuous and (con, dis) continuous–discrete sectors and finally (iii) (con, con) continuous–continuous sector with characters $\chi(p, \psi) = \exp i p \psi$; $p \in \mathbf{C}$. Recall that, due to torsion, the algebraic structure of the Dp branes wrapping the NC manifold change. Brane points $\{x_i\}$ of commutative type IIA geometry become fibers based on $\{x_i\}$. These fibers are valued in the algebra of the group representation $\mathcal{R}(C^{*r})$ and may be given a simple graph description. While points $x_i.1$ in the commutative type IIA geometry are essentially numbers, the Z_i coordinate operators can be thought of as

$$x_i.1 \rightarrow Z_i = (Z_i)_{mn} U^m V^n. \quad (39)$$

Extending the results of [24], one can draw graphs for fractional D branes. Due to the decomposition of I_{id} Eqs. (11) and (19), we associate to each Dp brane coordinate a quiver diagram mainly given by the product of (discrete or continuous) S^1 circles. For the simplest case $r = 1$ and C^* discrete representations, the quiver diagram is built as follows. (1) To each π_n projector it is associated a vertex point on a discrete S^1 circle. As there is an infinite number of points that one should put on S^1 , all happens as if the quiver diagram is given by the $\mathbf{Z} + i\mathbf{Z}$ lattice plus a extra point at infinity; (2) The a_n^{\pm} shift operators are associated with the oriented links joining adjacent vertices (vertex $(n - 1)$ to the vertex (n) for a_n^- and vertex (n) to the vertex $(n + 1)$ for a_n^+) of quiver diagram. They act as automorphisms exchanging the \mathbf{C}^* characters.

³ The sums involved in the decomposition of the identity are either discrete series, integrals or both of them depending on whether the group representation $\mathcal{R}(C^{*r})$ spectrum is discrete, continuous or with discrete and continuous sectors. Therefore, we have either $I_{\text{id}} = \sum_{\mathbf{n}} \pi_{\mathbf{n}}$, $\mathbf{n} = (n_1, \dots, n_r)$; $I_{\text{id}} = \int d\sigma \pi(\sigma)$, $\sigma = (\sigma_1, \dots, \sigma_r)$; or again $\sum_{\mathbf{m}} \int d\zeta \pi_{\mathbf{m}}(\zeta)$; Here $\pi_{\mathbf{m}}$, $\mathbf{m} = (m_1, \dots, m_{ir_0})$ and $\pi(\zeta)$, $\zeta = (\zeta_{ir_0+1}, \dots, \zeta_{ir})$; and are the C^{*r} representation projectors considered in Section 3.

Moreover, as D_p brane coordinates at the singularities are of the form $Z_i \sim \sum_n Z_i^{(n)}$, it follows that D_p branes on S sub-manifolds fractionate into an infinite set of fractional $D2_s$ branes coordinated by $Z_i^{(n)}$. This is a remarkable feature which looks like the process of tachyon condensation mechanism à la GMS [8,10–12] where for instance a noncompact D_{p1} brane on a NC Moyal plane decomposes into an infinite set of noncompact $D(p_1 - 2)$ branes. Such discussion is also valid for the case of a continuous spectrum; the corresponding quiver diagram is given by cross products of circles.

6. Conclusion

Using the BL algebraic geometry approach, we have studied the type IIA geometry of NCCY manifolds embedded in toric varieties \mathcal{V} endowed with a NC toric fibration. Actually this study completes partial results of works in the literature on NCCY manifolds and too particularly orbifolds of CY homogeneous hypersurfaces with discrete torsion. The results of this Letter concerns a more general class of singular CY manifolds \mathcal{M}_d embedded in toric varieties \mathcal{V}_{d+1} with \mathbf{C}^{*r} toric actions endowed by torsions. These torsions are carried either by quantum symmetries described by inner automorphisms of \mathbf{C}^{*r} or again by considering NC complex cycles within the toric group factors in the same manner as one does in the Connes et al. solution for toroidal compactification of matrix model of M-theory. Among our results, we find that complex d dimension NCCY manifolds $\mathcal{M}_d^{\text{nc}}$ are noncompact manifold that are naturally described in the language of toric geometry. The θ_{ij} parameters carrying the structure have contributions involving, in addition to the usual CY condition $\sum_{i=0}^k q_i^a = 0$, the data of the toric polygons using the relations $\sum_{i=0}^k q_i^a v_i^A = 0$.

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